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PROOF OF TWO CONJECTURES OF Z.-W. SUN

GUO-SHUAI MAO AND TAO ZHANG

ABSTRACT. In this paper, we prove two conjectures of Z.-W. Sun,

$$2n \binom{2n}{n} \left| \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 \right| 16^{n-1-k} \text{ for all } n = 2, 3, \dots,$$

and

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p + 2 \left(\frac{-1}{p} \right) p^3 E_{p-3} \pmod{p^4}$$

where $p > 3$ is a prime and E_0, E_1, E_2, \dots are Euler numbers.

1. INTRODUCTION

Let $p > 3$ be a prime. In 2011, Z.-W. Sun [Su1] investigated supercongruences and Euler numbers. He proved that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} &\equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} &\equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} &\equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}, \end{aligned}$$

and proposed the following conjectures.

Conjecture 1.1. [Su1, Conjecture 5.1].

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(i) For each $n = 2, 3, \dots$ we have

$$\begin{aligned} 2n \binom{2n}{n} &\left| \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 16^{n-1-k}, \right. \\ 2n \binom{2n}{n} &\left| \sum_{k=0}^{n-1} (6k+1) \binom{2k}{k}^3 256^{n-1-k}, \right. \\ 2n \binom{2n}{n} &\left| \sum_{k=0}^{n-1} (42k+5) \binom{2k}{k}^3 4096^{n-1-k}, \right. \end{aligned}$$

(ii) Let $p > 3$ be a prime. Then

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 &\equiv p + 2 \left(\frac{-1}{p} \right) p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} \frac{6k+1}{256^k} \binom{2k}{k}^3 &\equiv p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}, \\ \sum_{k=0}^{p-1} \frac{42k+5}{4096^k} \binom{2k}{k}^3 &\equiv 5p \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}, \end{aligned}$$

where $\left(\frac{\cdot}{p} \right)$ denotes the Legendre symbol.

We know the congruence conjecture

$$\sum_{k=0}^{p-1} \frac{42k+5}{4096^k} \binom{2k}{k}^3 \equiv 5 \left(\frac{-1}{p} \right) - p^3 E_{p-3} \pmod{p^4}$$

was solved by D.-W. Hu and G.-S. Mao [HM]. Many of other conjectures in [Su1, Conjecture 5.1] remain open.

In [Su2], Z.-W. Sun proved some products and sums divisible by central binomial coefficients, like

$$2(2n+1) \binom{2n}{n} \left| \binom{6n}{3n} \binom{3n}{n} \right| \text{ for all } n = 1, 2, 3, \dots$$

For any positive integer n ,

$$4(2n+1) \binom{2n}{n} \left| \sum_{k=0}^n (4k+1) \binom{2k}{k}^3 (-64)^{n-k} \right|$$

and

$$4(2n+1) \binom{2n}{n} \left| \sum_{k=0}^n (20k+3) \binom{2k}{k}^2 \binom{4k}{2k} (-2^{10})^{n-k} \right|$$

Victor J.W. Guo [G1] also did some conjectures of Z.-W. Sun on the divisibility problems, such as

$$(2n-1) \binom{3n}{n} \Big| \sum_{k=0}^n \binom{6k}{3k} \binom{3k}{k} \binom{6(n-k)}{3(n-k)} \binom{3(n-k)}{n-k},$$

and in [G2] he also showed that

$$(2n+3) \binom{2n}{n} \Big| 3 \binom{6n}{3n} \binom{3n}{n} \text{ and } (10n+3) \binom{3n}{n} \Big| 21 \binom{15n}{5n} \binom{5n}{n}.$$

Motivated by the above work, we obtain the following result.

Theorem 1.1. *For any integer $n = 2, 3, 4, \dots$, we have*

$$2n \binom{2n}{n} \Big| \sum_{k=0}^{n-1} (3k+1) \binom{2k}{k}^3 16^{n-1-k}. \quad (1.1)$$

We will prove Theorem 1.1 in Section 2.

In 2012, J. Guillera and W. Zudilin [WZ] proved that

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p \pmod{p^3}$$

using Wilf-Zeilberger method.

Recently, G.-S. Mao and Z.-W. Sun [MS] proved that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k \equiv \frac{2}{3} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \pmod{p}.$$

Combine them together, we obtain the following result.

Theorem 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 \equiv p + 2 \left(\frac{-1}{p} \right) p^3 E_{p-3} \pmod{p^4}, \quad (1.2)$$

where $H_n := \sum_{k=1}^n (n = 1, 2, \dots)$ denote the Harmonic numbers.

We will prove Theorem 1.2 in Section 3.

2. PROOF OF THEOREM 1.1

Lemma 2.1. [Kummer, 1852] *The p -adic valuation of the binomial coefficient $\binom{m+n}{m}$ is equal to the number of 'carry-overs' when performing the addition of n and m , written in base p .*

Lemma 2.2. *For any positive integer $n \geq 6$ and $n \neq 2^m + 1$, where m is an integer, we have*

$$n - \text{ord}_2((n-1)!) \geq 3,$$

where $\text{ord}_2((n-1)!) = \sum_{i=1}^{\infty} \lfloor \frac{n-1}{2^i} \rfloor$.

Proof. Set $n = a_1 a_2 \cdots a_k$, where $a_1 = 1$, and $a_i \in \{0, 1\}$ for all $2 \leq i \leq k$, $n \geq 6$, so we have $k > 2$.

Case 1. $2 \mid n$, so $a_k = 0$, if $a_{k-1} = 0$, then $4 \mid n$, and we know $2 \mid \binom{2n}{n}$, thus $8 \mid n \binom{2n}{n}$. else if $a_{k-1} = 1$, then $2 \mid n$ and $4 \nmid n$, but we know $k > 2$, and $a_1 = 1$, so by Kummer's Theorem we can get that $4 \mid \binom{2n}{n}$, hence $8 \mid n \binom{2n}{n}$.

Case 2. $2 \nmid n$, so $a_k = 1$, we have $k > 2$ and $a_1 = 1$, but $n \neq 2^m + 1$, so there exists an $2 \leq i \leq k-1$ such that $a_i = 1$, so by Kummer's Theorem we can deduce that $8 \mid \binom{2n}{n}$, thus $8 \mid n \binom{2n}{n}$.

So we have $\text{ord}_2(n \binom{2n}{n}) \geq 3$, we know $n \binom{2n}{n} = \frac{2^n (2n-1)!!}{(n-1)!}$, so

$$\text{ord}_2(n \binom{2n}{n}) = n - \text{ord}_2((n-1)!),$$

therefore

$$n - \text{ord}_2((n-1)!) \geq 3.$$

So we have done the proof of Lemma 2.2.

Lemma 2.3. *For any positive integer $n \geq 6$ and $n = 2^m + 1$, where m is an integer, then for any $1 \leq k \leq n-2$ we have*

$$2 \mid \binom{n-1}{k}.$$

Proof. We know $\binom{n-1}{k} = \binom{2^m}{k} = \frac{2^m}{k} \binom{2^m-1}{k-1}$, so $\text{ord}_2(\binom{2^m}{k}) \geq m - \text{ord}_2(k) \geq 1$ for all $1 \leq k \leq n-2$, so we have $\text{ord}_2(\binom{n-1}{k}) \geq 1$ for all $1 \leq k \leq n-2$. Thus we get the desired result.

Lemma 2.4. *For any positive integer $n \geq 6$ and $n = 2^m + 1$, where m is an integer, then we have*

$$8 \mid \binom{4n-2}{2n-1} + 2 \binom{2n-2}{n-1}.$$

Proof. First we know

$$\begin{aligned} \binom{4n-2}{2n-1} &= \frac{(4n-2)(4n-3) \cdots (2n+1)(2n)!}{(2n-1)^2(2n-2)^2 \cdots (n+1)^2(n!)^2} = \frac{2^n(4n-3)!!}{(2n-1)!!(n-1)!} \\ &= \frac{2^n(4n-3)(4n-5) \cdots (2n+1)}{(n-1)!} \end{aligned}$$

hence

$$\begin{aligned}
& \binom{4n-2}{2n-1} + 2 \binom{2n-2}{n-1} \\
&= \frac{2^n(4n-3)(4n-5) \cdots (2n+1)}{(n-1)!} + \frac{2^n(2n-3)!!}{(n-1)!} \\
&= \frac{2^n}{(n-1)!} ((4n-3)(4n-5) \cdots (2n+1) + (2n-3)!!).
\end{aligned}$$

We can see that $(4n-3)(4n-5) \cdots (2n+1) + (2n-3)!! \equiv 0 \pmod{2}$. So we can deduce that

$$\text{ord}_2\left(\binom{4n-2}{2n-1} + 2 \binom{2n-2}{n-1}\right) \geq n+1 - \text{ord}_2((n-1)!) = n+1 - \text{ord}_2(n!),$$

since $n = 2^m + 1$ is an odd integer. While $n \geq 6$ and $n = 2^m + 1$ we have

$$\begin{aligned}
n+1 - \text{ord}_2(n!) &= 2^m + 2 - \sum_{i=1}^{\infty} \left\lfloor \frac{2^m + 1}{2^i} \right\rfloor \\
&= 2^m + 2 - (2^{m-1} + 2^{m-2} + \cdots + 2 + 1) = 2^m + 2 - (2^m - 1) = 3.
\end{aligned}$$

Therefore $\text{ord}_2\left(\binom{4n-2}{2n-1} + 2 \binom{2n-2}{n-1}\right) \geq 3$, that is $8 \mid \binom{4n-2}{2n-1} + 2 \binom{2n-2}{n-1}$, so we finish the proof of Lemma 2.4.

Lemma 2.5. [MS, Lemma3.2] *For any nonnegative integer n , we have*

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{x+k}{2n+1} = \frac{1}{(4n+2) \binom{2n}{n}} \sum_{k=0}^n (2x-3k) \binom{x}{k}^2 \binom{2k}{k}.$$

Proof of Theorem 1.1. For all $n = 2, 3, \dots$, replacing n by $n-1$ and $x = -\frac{1}{2}$ in Lemma 2.5, we can deduce that

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{-\frac{1}{2}+k}{2n-1} = \frac{1}{(4n-2) \binom{2n-2}{n-1}} \sum_{k=0}^{n-1} (-1-3k) \binom{-\frac{1}{2}}{k}^2 \binom{2k}{k}.$$

We know the right side is $\frac{-1}{n \binom{2n}{n}} \sum_{k=0}^{n-1} (3k+1) \frac{\binom{2k}{k}^3}{16^k}$. So we want to prove Theorem 1.1, we just need to show

$$\frac{16^{n-1}}{2} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{-\frac{1}{2}+k}{2n-1} \in \mathbb{Z}.$$

While we know that

$$\begin{aligned} & \frac{16^{n-1}}{2} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \left(-\frac{1}{2} + k\right) \\ &= \frac{1}{8} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1} (2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!}, \end{aligned}$$

so we just need to show

$$\frac{1}{8} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1} (2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!} \in \mathbb{Z}.$$

For any real numbers x and y , we have (see, for example, [PS, Division 8, Problems 8 and 136])

$$\begin{aligned} \lfloor 2x \rfloor + \lfloor 2y \rfloor &\geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor, \\ \lfloor x + y \rfloor &\geq \lfloor x \rfloor + \lfloor y \rfloor. \end{aligned}$$

So for any integer $m > 1$ we have

$$\left\lfloor \frac{2k}{m} \right\rfloor + \left\lfloor \frac{4n-2k-2}{m} \right\rfloor \geq \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \frac{2n-1}{m} \right\rfloor + \left\lfloor \frac{2n-k-1}{m} \right\rfloor,$$

that is to say for any prime p we have

$$\sum_{i=1}^{\infty} \left(\left\lfloor \frac{2k}{p^i} \right\rfloor + \left\lfloor \frac{4n-2k-2}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{2n-1}{p^i} \right\rfloor - \left\lfloor \frac{2n-k-1}{p^i} \right\rfloor \right) \geq 0,$$

hence

$$\frac{(-1)^{k+1} (2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!} \in \mathbb{Z}.$$

While

$$\frac{(2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!} = \frac{2^n (2k-1)! (4n-2k-3)!}{(2n-1)! (n-1)!},$$

so

$$\text{ord}_2 \left(\frac{(2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!} \right) = n - \text{ord}_2((n-1)!) \geq 1$$

for all integer $n \geq 2$, then

$$2 \mid a(n, k) := \frac{(2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!}. \quad (2.1)$$

With Lemma 2.2 we have $8 \mid \frac{(2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!}$ for all integer $n \geq 6$ and $n \neq 2^m + 1$, where m is an integer.

With Lemma 2.3 and (2.1), we have $8 \mid \binom{n-1}{k}^2 \frac{(2k)! (4n-2k-2)!}{k! (2n-1)! (2n-k-1)!}$ for all $1 \leq k \leq n-2$, where $n = 2^m + 1 \geq 6$ and m is an integer.

With Lemma 2.4 we know when $k = 0$ and $k = n - 1$, we have $8 \mid a(n, 0) + a(n, n - 1)$ for all integer $n = 2^m + 1 \geq 6$, where m is an integer.

So combining these we can deduce that for all integer $n \geq 6$ we have

$$8 \mid \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1}(2k)!(4n-2k-2)!}{k!(2n-1)!(2n-k-1)!},$$

that is to say

$$\frac{1}{8} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1}(2k)!(4n-2k-2)!}{k!(2n-1)!(2n-k-1)!} \in \mathbb{Z}$$

for all integer $n \geq 6$.

When $n = 2, 3, 4, 5$, we have

$$\sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1}(2k)!(4n-2k-2)!}{k!(2n-1)!(2n-k-1)!}$$

equals to $-16, -152, -1664, -20072$ respectively, we can easily check that $-16, -152, -1664, -20072$ are divisible by 8.

So combining these we can finally get that for all $n = 2, 3, 4, \dots$, we have

$$\frac{1}{8} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \frac{(-1)^{k+1}(2k)!(4n-2k-2)!}{k!(2n-1)!(2n-k-1)!} \in \mathbb{Z}.$$

Then we complete the proof of Theorem 1.1. \square

3. PROOF OF THEOREM 1.2

Lemma 3.1 (Staver [S]). *For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=1}^n \frac{\binom{2k}{k}}{k} = \frac{n+1}{3} \binom{2n+1}{n} \sum_{k=1}^n \frac{1}{k^2 \binom{n}{k}^2} \quad (3.1)$$

Lemma 3.2 (Morley [M]). *Let $p > 3$ be a prime. Then*

$$\left(\binom{p-1}{(p-1)/2} \right) \equiv \left(\frac{-1}{p} \right) 4^{p-1} \pmod{p^3}. \quad (3.2)$$

Lemma 3.3. *Let $p > 3$ be a prime. For any $0 < k \leq (p-1)/2$, we have*

$$\frac{1}{p} \binom{p-1+2k}{(p-1)/2+k} \equiv \left(\frac{-1}{p} \right) 4^{p-1} \frac{4^{2k}}{2k \binom{2k}{k}} (1 - p(H_{2k-1} - H_{k-1})) \pmod{p^2}. \quad (3.3)$$

In particular,

$$\frac{1}{p} \binom{p-1+2k}{(p-1)/2+k} \equiv \left(\frac{-1}{p} \right) \frac{4^{2k}}{2k \binom{2k}{k}} \pmod{p}. \quad (3.4)$$

Proof. Expand the LHS and use the Morley congruence (3.2), we have

$$\begin{aligned} \frac{1}{p} \binom{p-1+2k}{(p-1)/2+k} &= \binom{p-1}{(p-1)/2} \frac{(p+1) \cdots (p+2k-1)}{(\prod_{j=1}^k (p+2j-1)/2)^2} \\ &\equiv \left(\frac{-1}{p} \right) 4^{p-1} \frac{(2k-1)!(1+pH_{2k-1})2^{2k}}{\prod_{j=1}^k (p+2j-1)^2} \\ &\equiv \left(\frac{-1}{p} \right) 4^{p-1} \frac{(2k-1)!(1+pH_{2k-1})2^{2k}}{\prod_{j=1}^k ((2j-1)^2 + 2p(2j-1))} \\ &\equiv \left(\frac{-1}{p} \right) 4^{p-1} \frac{(2k-1)!(1+pH_{2k-1})2^{2k}}{((2k-1)!!)^2 (1+2p \sum_{j=1}^k 1/(2j-1))} \\ &= \left(\frac{-1}{p} \right) \frac{4^{p-1+2k} (1+pH_{2k-1})}{2k \binom{2k}{k} (1+2p(H_{2k-1} - H_{k-1}/2))} \pmod{p^2}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{1+2p(H_{2k-1} - H_{k-1}/2)} &= \frac{1-2p(H_{2k-1} - H_{k-1}/2)}{1-4p^2(H_{2k-1} - H_{k-1}/2)^2} \\ &\equiv 1-2p(H_{2k-1} - H_{k-1}/2) \pmod{p^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{p} \binom{p-1+2k}{(p-1)/2+k} &\equiv \left(\frac{-1}{p} \right) \frac{4^{p-1+2k}}{2k \binom{2k}{k}} (1+pH_{2k-1})(1-2p(H_{2k-1} - H_{k-1}/2)) \\ &\equiv \left(\frac{-1}{p} \right) \frac{4^{p-1+2k}}{2k \binom{2k}{k}} (1-p(H_{2k-1} - H_{k-1})) \pmod{p^2}, \end{aligned}$$

as desired. The last statment follows immediately from Fermat's Little Theorem. \square

Lemma 3.4. *Let $p > 3$ be a prime. For any $0 < k \leq (p-1)/2$, we have*

$$\frac{1}{p} \binom{p-1+2k}{2k} \equiv \frac{1}{2k} (1+pH_{2k-1}) \pmod{p^2} \quad (3.5)$$

In particular,

$$\frac{1}{p} \binom{p-1+2k}{2k} \equiv \frac{1}{2k} \pmod{p}. \quad (3.6)$$

Proof. Expand the LHS, we have

$$\begin{aligned}
\frac{1}{p} \binom{p-1+2k}{2k} &= \frac{(2k+1) \cdots (p-1)}{1 \cdots (p-2k-1)} \cdot \frac{1}{p-2k} \cdot \frac{(p+1) \cdots (p+2k-2)}{(p-2k+1) \cdots (p-1)} \\
&= \frac{1}{2k} \prod_{j=1}^{p-2k-1} (1 - p/j) \cdot \frac{1}{1 - p/2k} \cdot \prod_{j=1}^{2k-1} \frac{1 + p/j}{1 - p/j} \\
&\equiv \frac{1}{2k} (1 - pH_{p-1-2k})(1 + p/2k)(1 + 2pH_{2k-1}) \\
&\equiv \frac{1}{2k} (1 - pH_{p-1-2k} + p/2k + 2pH_{2k-1}) \pmod{p^2}.
\end{aligned}$$

Recall that Wolstenholem [W] proved that for any prime $p > 3$,

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

It follows that

$$\begin{aligned}
H_{p-2k-1} &\equiv -H_{p-1} + H_{p-2k-1} = -\sum_{j=1}^{2k} 1/(p-j) \\
&\equiv -\sum_{j=1}^{2k} 1/(-j) = H_{2k} \pmod{p}.
\end{aligned}$$

Therefore,

$$\frac{1}{p} \binom{p-1+2k}{2k} \equiv \frac{1}{2k} (1 + pH_{2k-1}) \pmod{p^2}.$$

□

Lemma 3.5. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{4^{2k}}{k^2 \binom{2k}{k}^2} \equiv (-1)^{(p-1)/2} \frac{3}{p} 4^{1-p} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \pmod{p} \quad (3.7)$$

Proof. Note that

$$\binom{(p-2)/1}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}.$$

It follows that

$$\sum_{k=1}^{(p-1)/2} \frac{4^{2k}}{k^2 \binom{2k}{k}^2} = \sum_{k=1}^{(p-1)/2} \frac{(-4)^{2k}}{k^2 \binom{2k}{k}^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{p-1/2}{k}^2} \pmod{p}.$$

By Staver identity (3.1), we have

$$\begin{aligned}
\sum_{k=1}^{(p-1)/2} \frac{4^{2k}}{k^2 \binom{2k}{k}^2} &\equiv \frac{3}{\frac{p+1}{2} \binom{p}{(p-1)/2}} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \\
&= \frac{3}{\frac{p+1}{2} \frac{p}{(p+1)/2} \binom{p-1}{(p-1)/2}} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \\
&\equiv (-1)^{(p-1)/2} \frac{3}{p} 4^{1-p} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \pmod{p},
\end{aligned}$$

where we use Morley congruence (3.2) in the last step.

Proof of Theorem 1.2. Take the same WZ pair $F(k, j)$ and $G(k, j)$ as in [WZ],

$$\begin{aligned}
F(k, j) &= \frac{2k+2j+1}{16^k} \binom{2k}{k}^2 \frac{\binom{2k+2j}{k+j} \binom{2k+2j}{2j}}{\binom{2j}{j}}, \\
G(k, j) &= -\frac{2(2k-1)}{16^{k-1}} \binom{2k-2}{k-1}^2 \frac{\binom{2k+2j-2}{k+j-1} \binom{2k+2j-2}{2j}}{\binom{2j}{j}}.
\end{aligned}$$

We know that $F(k, j)$ and $G(k, j)$ have the following relation,

$$F(k, j-1) - F(k, j) = G(k+1, j) - G(k, j).$$

Sum the above equation, first from $k = 0$ to $k = (p-1)/2$, then from $j = 1$ to $(p-1)/2$, we get

$$\sum_{k=0}^{(p-1)/2} (F(k, 0) - F(k, (p-1)/2)) = \sum_{j=1}^{(p-1)/2} (G((p+1)/2, j) - G(0, j)).$$

Note that $G(0, j) = 0$ and

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 = \sum_{k=0}^{(p-1)/2} F(k, 0),$$

we have

$$\sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 = \sum_{k=0}^{(p-1)/2} F(k, (p-1)/2) + \sum_{j=1}^{(p-1)/2} G((p+1)/2, j).$$

Hence, it suffices to determine

$$\sum_{k=0}^{(p-1)/2} F(k, (p-1)/2) \text{ and } \sum_{j=1}^{(p-1)/2} G((p+1)/2, j) \pmod{p^4}.$$

First, let's consider

$$\begin{aligned} G((p+1)/2, j) &= -\frac{2p}{4^{p-1}} \binom{p-1}{(p-1)/2}^2 \frac{\binom{p-1+2j}{(p-1)/2+j} \binom{p-1+2j}{2j}}{\binom{2j}{j}} \\ &= -\frac{2p^3}{4^{p-1}} \cdot \frac{\binom{p-1}{(p-1)/2}^2}{\binom{2j}{j}} \cdot \frac{\binom{p-1+2j}{(p-1)/2+j}}{p} \cdot \frac{\binom{p-1+2j}{2j}}{p}. \end{aligned}$$

Apply Lemma 3.2, 3.3, 3.4 and 3.5, we get

$$\sum_{j=1}^{(p-1)/2} G((p+1)/2, j) \equiv -\frac{3}{2}p^2 \sum_{j=1}^{(p-1)/2} \frac{\binom{2j}{j}}{j} \pmod{p^4}.$$

Now, let's consider

$$F(k, (p-1)/2) = \frac{3k+p}{16^k} \binom{2k}{k}^2 \frac{\binom{2k+p-1}{k+(p-1)/2} \binom{2k+p-1}{2k}}{\binom{p-1}{(p-1)/2}}.$$

Apply Lemma 3.2, 3.3 and 3.4

$$\begin{aligned} &\sum_{k=0}^{(p-1)/2} F\left(k, \frac{p-1}{2}\right) \\ &= p + \sum_{k=1}^{(p-1)/2} \frac{p}{16^k} \binom{2k}{k}^2 \frac{\binom{2k+p-1}{k+(p-1)/2} \binom{2k+p-1}{2k}}{\binom{p-1}{(p-1)/2}} + \sum_{k=1}^{(p-1)/2} \frac{3k}{16^k} \binom{2k}{k}^2 \frac{\binom{2k+p-1}{k+(p-1)/2} \binom{2k+p-1}{2k}}{\binom{p-1}{(p-1)/2}} \\ &\equiv p + \frac{p^3}{4} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k^2} + \frac{3p^2}{4} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} (1 - pH_{2k-1} + pH_{k-1})(1 + pH_{2k-1}) \\ &\equiv p + \frac{p^3}{4} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k^2} + \frac{3p^2}{4} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} (1 + pH_{k-1}) \pmod{p^4}. \end{aligned}$$

Combine all these above together, we have

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{3k+1}{16^k} \binom{2k}{k}^3 &\equiv p - \frac{3}{4}p^2 \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} - \frac{1}{2}p^3 \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k^2} \\ &\quad + \frac{3}{4}p^3 \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} H_k \pmod{p^4}. \end{aligned}$$

Fianlly, apply the congruence ([Su1])

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2},$$

and the congruence ([MS])

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k \equiv \frac{2}{3} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \pmod{p}.$$

we get the desired result.

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(GUO-SHUAI MAO) DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY,
NANJING 210093, PEOPLE'S REPUBLIC OF CHINA

E-mail address: mg1421007@smail.nju.edu.cn

(TAO ZHANG) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND,
COLLEGE PARK, MD 20742, USA

E-mail address: taozhang@math.umd.edu